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ADDENDUM

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Abstract. The normally ordered Hilbert space image of the complex linear similarity transformation in Grassmann number phase space is derived in fermionic coherent state representation. Though keeping the anticommutator of Fermi operators invariant, the quantum mechanical images of classical transformations are generally not unitary. Remarkably, although kets and bras produced by the non-unitary similarity transformations are not Hermitian conjugates, they still form a complete basis. The derivation of the normally ordered operator which engenders the similarity transformation is facilitated by virtue of the technique of integration within order products.

1. Introduction

In a previous work [1], the normally ordered Hilbert space image of the general (complex) linear similarity in phase space was obtained in coherent state representation. Although preserving the bosonic commutator of Bose operators, these quantum mechanical images of classical transformations are generally not unitary. Remarkably, we have demonstrated in [1] that although the kets and bras produced by the non-unitary similarity transformations are not Hermitian conjugates, squeezed states analogues produced using the similarity transformation still satisfy an overcompleteness relation. In this work we continue to discuss fermionic case. We shall show that the kets and bras, which are produced by the non-unitary fermionic similarity transformation and still satisfy an overcompleteness relation, can also be found. We use the fermionic coherent state rerpesentation, and the evaluation of the normally ordered Hilbert space image of the complex linear similarity transformation in Grassmann number space is performed by the use of the technique of integration within ordered products (IWOP). In section 2 we investigate the general linear similarity transformation of the Fermi creation operators $a^\dagger$, $b^\dagger$ and annihilation operators a, b given by

$$\begin{aligned} d &\equiv VaV^{-1} = \nu b^\dagger - \mu a \qquad & g^\dagger &\equiv Va^\dagger V^{-1} = \sigma b - \tau a^\dagger \\ h &\equiv VbV^{-1} = -\mu b - \nu a^\dagger \qquad & k^\dagger &\equiv Vb^\dagger V^{-1} = -\sigma a - \tau b^\dagger \end{aligned} \tag{1}$$

with μ, ν, σ, τ arbitrary complex numbers satisfying $\mu\tau + \sigma\nu = 1$ (note that this condition is different from the requirement for bosonic system). By virtue of the IWOP technique we derive the normally ordered similarity transformation operator V and show that V preserves the anti-commutator $\{d, g^\dagger\} = \{k^\dagger, h\} = 1$, $\{g^\dagger, k^\dagger\} = \{d, h\} = 0$ even though d and $g^\dagger$ (h and $k^\dagger$) are not generally Hermitian conjugates. In section 3 we construct the kets and bras in coherent state representation engendered by the similarity transformation to make up overcompleteness relation. In section 4 we give an example showing the utility of the fermionic similarity transformation.

2. Derivation of V in the fermionic coherent state representation

Enlightened by the earlier works [1, 2] we postulate the following fermionic coherent state representation for V:

$$V=-\tau^{-1}\int \mathrm{d}\bar{\alpha}_1\,\mathrm{d}\alpha_1\int \mathrm{d}\bar{\alpha}_2\,\mathrm{d}\alpha_2\left|\left(\begin{matrix}-\tau & 0 & 0 & -\nu\\ 0 & -\mu & -\sigma & 0\\ 0 & \nu & -\tau & 0\\ \sigma & 0 & 0 & -\mu\end{matrix}\right.\begin{matrix}\alpha_1\\ \bar{\alpha}_1\\ \alpha_2\\ \bar{\alpha}_2\end{matrix}\right)\right\rangle\left\langle\left(\begin{matrix}\alpha_1\\ \bar{\alpha}_1\\ \alpha_2\\ \bar{\alpha}_2\end{matrix}\right)\right| \tag{2}$$

where the bra is the fermionic coherent state [3] defined as

$$\begin{aligned}\left\langle\left(\begin{matrix}\alpha_1\\ \bar{\alpha}_1\\ \alpha_2\\ \bar{\alpha}_2\end{matrix}\right)\right| &\equiv\langle\alpha_1\alpha_2|\\ &=\langle 00|\exp[-a^{\dagger}\alpha_1+\bar{\alpha}_1 a-b^{\dagger}\alpha_2+\bar{\alpha}_2 b]\\ &=\langle 00|\exp[-\tfrac{1}{2}(\bar{\alpha}_1\alpha_1+\bar{\alpha}_2\alpha_2)+\bar{\alpha}_1 a+\bar{\alpha}_2 b]\end{aligned} \tag{3}$$

and the ket defined as

$$\begin{aligned}&\left|\left(\begin{matrix}-\tau & 0 & 0 & -\nu\\ 0 & -\mu & -\sigma & 0\\ 0 & \nu & -\tau & 0\\ \sigma & 0 & 0 & -\mu\end{matrix}\right)\left(\begin{matrix}\alpha_1\\ \bar{\alpha}_1\\ \alpha_2\\ \bar{\alpha}_2\end{matrix}\right)\right\rangle=\left|\left(\begin{matrix}-\tau\alpha_1-\nu\bar{\alpha}_2\\ -\mu\bar{\alpha}_1-\sigma\alpha_2\\ \nu\bar{\alpha}_1-\tau\alpha_2\\ \sigma\alpha_1-\mu\bar{\alpha}_2\end{matrix}\right)\right\rangle\\ &\quad=\exp[a^{\dagger}(-\tau\alpha_1-\nu\bar{\alpha}_2)-(-\mu\bar{\alpha}_1-\sigma\alpha_2)a+b^{\dagger}(\nu\bar{\alpha}_1-\tau\alpha_2)\\ &\qquad-(\sigma\alpha_1-\mu\bar{\alpha}_2)b]|00\rangle\end{aligned} \tag{4}$$

$$\begin{aligned}&=\exp[-\tfrac{1}{2}(-\mu\bar{\alpha}_1-\sigma\alpha_2)(-\tau\alpha_1-\nu\bar{\alpha}_2)+a^{\dagger}(-\tau\alpha_1-\nu\bar{\alpha}_2)\\ &\quad-\tfrac{1}{2}(\sigma\alpha_1-\mu\bar{\alpha}_2)(\nu\bar{\alpha}_1-\tau\alpha_2)+b^{\dagger}(\nu\bar{\alpha}_1-\tau\alpha_2)]|00\rangle.\end{aligned} \tag{4'}$$

Note that in contrast to the definition of the usual fermionic coherent state, in (4) the coefficients of Fermi operators are not complex conjugates. The Grassmann numbers in (2) obey the rules:

$$\{\alpha_i,a\}=\{\alpha_i,b\}=0\qquad\{\alpha_i,\bar{\alpha}_j\}=0\qquad\alpha_i^2=\bar{\alpha}_i^2=0\qquad(i,j)=1,2 \tag{5}$$

$$\int\mathrm{d}\alpha_i=0\qquad\int\mathrm{d}\alpha_i\alpha_i=1 \tag{6}$$

and the two-mode vacuum state projection operator $|00\rangle\langle 00|$ can be expressed as

$$|00\rangle\langle 00|=:\exp[-a^{\dagger}a-b^{\dagger}b]: \tag{7}$$

where : : stands for the normal product. By virtue of the IWOP technique for fermionic system [2] (note that Fermi operators are anti-permuted within : :) and the mathematical formula

$$\int \prod_i \mathrm{d}\bar{\alpha}_i \, \mathrm{d}\alpha_i \exp\left\{-\sum_{i,j} \bar{\alpha}_i \Lambda_{ij} \alpha_j + \sum_i (\bar{\alpha}_i \eta_i + \bar{\eta}_i \alpha_i)\right\} = (\det \Lambda) \exp\left[\sum_{ij} \bar{\eta}_i (\Lambda^{-1})_{ij} \eta_j\right] \tag{8}$$

where η_i and $\bar{\eta}_i$ are also Grassmann numbers, while Λ is a complex matrix, we perform the integration of (2) and get the normally ordered form of V

$$\begin{aligned} V = -\tau^{-1} \int & \mathrm{d}\bar{\alpha}_1 \, \mathrm{d}\alpha_1 \int \mathrm{d}\bar{\alpha}_2 \, \mathrm{d}\alpha_2 \exp\{-\tfrac{1}{2}(\mu\tau - \sigma\nu)(\bar{\alpha}_1 \alpha_1 + \bar{\alpha}_2 \alpha_2) + \mu\nu\bar{\alpha}_2\bar{\alpha}_1 \\ & + \sigma\tau\alpha_1\alpha_2 + b^\dagger(\nu\bar{\alpha}_1 - \tau\alpha_2) - a^\dagger(\tau\alpha_1 + \nu\bar{\alpha}_2)\}|00\rangle\langle\alpha_1\alpha_2| \\ = -\tau^{-1} & \int \mathrm{d}\bar{\alpha}_1 \, \mathrm{d}\alpha_1 \int \mathrm{d}\bar{\alpha}_2 \, \mathrm{d}\alpha_2 :\exp\{-\mu\tau(\bar{\alpha}_1\alpha_1 + \bar{\alpha}_2\alpha_2) + \bar{\alpha}_2(\mu\nu\bar{\alpha}_1 + \nu a^\dagger + b) \\ & + (\sigma\tau\alpha_1 - \tau b^\dagger)\alpha_2 + \bar{\alpha}_1(a - \nu b^\dagger) - \tau a^\dagger \alpha_1 - a^\dagger a - b^\dagger b\}: \\ = -\mu & \exp\left(\frac{\nu}{\mu} a^\dagger b^\dagger\right) : \exp\left[-\left(\frac{1}{\mu} + 1\right)(a^\dagger a + b^\dagger b)\right] : \exp\left(\frac{\sigma}{\mu} ab\right). \end{aligned} \tag{9}$$

With the help of (9) and its inverse

$$V^{-1} = -\mu^{-1} \exp\left(-\frac{\sigma}{\mu} ab\right) \exp\left\{-\ln\left(-\frac{1}{\mu}\right)(a^\dagger a + b^\dagger b)\right\} \exp\left(-\frac{\nu}{\mu} a^\dagger b^\dagger\right) \tag{10}$$

we easily prove that V generates the transformation (1) as

$$\begin{aligned} V a V^{-1} &= \exp\left(\frac{\nu}{\mu} a^\dagger b^\dagger\right)(-\mu a) \exp\left(-\frac{\nu}{\mu} a^\dagger b^\dagger\right) = \nu b^\dagger - \mu a \equiv d \\ V b V^{-1} &= -\mu b - \nu a^\dagger \equiv h \end{aligned} \tag{11}$$

$$\begin{aligned} V a^\dagger V^{-1} &= \exp\left(\frac{\nu}{\mu} a^\dagger b^\dagger\right)\left(-\frac{a^\dagger}{\mu} + \sigma b\right) \exp\left(-\frac{\nu}{\mu} ab\right) = \sigma b - \tau a^\dagger \equiv g^\dagger \\ V b^\dagger V^{-1} &= -\sigma a - \tau b^\dagger \equiv k^\dagger. \end{aligned} \tag{12}$$

The inverse transformations follow immediately from (10) and (9)

$$\begin{aligned} & V^{-1} a V = \exp\left(-\frac{\sigma}{\mu} ab\right)\left(-\frac{a}{\mu} - \nu b^\dagger\right) \exp\left(\frac{\sigma}{\mu} ab\right) = -\tau a - \nu b^\dagger \\ & V^{-1} b V = -\tau b + \nu a^\dagger \\ & V^{-1} a^\dagger V = -\mu a^\dagger - \sigma b \qquad V^{-1} b^\dagger V = -\mu b^\dagger + \sigma a. \end{aligned} \tag{13}$$

We will also require V^{-1} in normal order for later uses. Again using the IWOP technique and the overcompleteness relation for the coherent state,

$$\begin{aligned} \int & \mathrm{d}\bar{\alpha}_1 \, \mathrm{d}\alpha_1 \int \mathrm{d}\bar{\alpha}_2 \, \mathrm{d}\alpha_2 |\alpha_1\alpha_2\rangle\langle\alpha_1\alpha_2| \\ & = \int \mathrm{d}\bar{\alpha}_1 \, \mathrm{d}\alpha_1 \int \mathrm{d}\bar{\alpha}_2 \, \mathrm{d}\alpha_2 \\ & \quad \times :\exp\left[-\sum_{i=1}^{2} (\bar{\alpha}_i - a_i^\dagger)(\alpha_i - a_i)\right]: = 1 \qquad (a_2 \equiv b) \end{aligned} \tag{14}$$

we calculate

$$V^{-1}=-\mu^{-1}\exp\left(-\frac{\sigma}{\mu}ab\right)\int d\bar{\alpha}_1\,d\alpha_1\int d\bar{\alpha}_2\,d\alpha_2$$
$$\times\exp[-\tfrac{1}{2}(\bar{\alpha}_1\alpha_1+\bar{\alpha}_2\alpha_2)-\mu(a^\dagger\alpha_1+b^\dagger\alpha_2)]|00\rangle$$
$$\times\langle\alpha_1\alpha_2|\exp\left(-\frac{\nu}{\mu}a^\dagger b^\dagger\right)$$
$$=-\mu^{-1}\int d\bar{\alpha}_1\,d\alpha_1\int d\bar{\alpha}_2\,d\alpha_2\,:\exp\left\{-\bar{\alpha}_1\alpha_1-\bar{\alpha}_2\alpha_2+\bar{\alpha}_1\left(a-\frac{\nu}{\mu}\bar{\alpha}_2\right)\right.$$
$$\left.+(\sigma\mu\alpha_2-\mu a^\dagger)\alpha_1+\bar{\alpha}_2 b-\mu b^\dagger\alpha_2-a^\dagger a-b^\dagger b\right\}:$$
$$=-\tau\exp\left[-\frac{\nu}{\tau}a^\dagger b^\dagger\right]:\exp\left[\left(-\frac{1}{\tau}-1\right)(a^\dagger a+b^\dagger b)\right]:\exp\left[-\frac{\sigma}{\tau}ab\right].\tag{15}$$

Comparing (15) with the Hermitian conjugate of (9) we see $V^{-1}\neq V^\dagger$, V is not unitary.

3. Kets and bras generated by V and V^{-1} which satisfy the completeness relation

In terms of the normally ordered form of V we construct the following state

$$D(\alpha_1)D(\alpha_2)V|00\rangle=-\mu\exp\left[\frac{\nu}{\mu}(a^\dagger-\bar{\alpha}_1)(b^\dagger-\bar{\alpha}_2)\right]|\alpha_1\alpha_2\rangle\equiv|\alpha_1\alpha_2;\mu,\nu\rangle\tag{16}$$

where $D(\alpha_1)=\exp(a^\dagger\alpha_1-\bar{\alpha}_1 a)$, $D(\alpha_2)=\exp(b^\dagger\alpha_2-\bar{\alpha}_2 b)$, α_1, α_2 are Gasssmann numbers. It is worth noting that $|\alpha_1,\alpha_2;\mu,\nu\rangle$ satisfies the eigenvalue equations

$$(\mu a-\nu b^\dagger)|\alpha_1,\alpha_2;\mu,\nu\rangle=(\mu\alpha_1-\nu\tilde{\alpha}_2)|\alpha_1,\alpha_2;\mu,\nu\rangle$$
$$(\mu b+\nu a^\dagger)|\alpha_1,\alpha_2;\mu,\nu\rangle=(\mu\alpha_2+\nu\tilde{\alpha}_1)|\alpha_1,\alpha_2;\mu,\nu\rangle.$$

In a similar fashion we construct $\langle 00|V^{-1}D^\dagger(\alpha_1)D^\dagger(\alpha_2)$, noting that as V is not unitary, V^{-1}, not $V^\dagger$, is employed to produce the bra

$$\langle 00|V^{-1}D^\dagger(\alpha_2)D^\dagger(\alpha_1)=-\tau\langle\alpha_1\alpha_2|\exp\left[-\frac{\sigma}{\tau}(a-\alpha_1)(b-\alpha_2)\right]\equiv\langle\alpha_1\alpha_2;\sigma,\tau|\tag{18}$$

which satisfies

$$\begin{aligned}\langle\alpha_1\alpha_2;\sigma,\tau|(\mu a^\dagger-\sigma b^\dagger)&=\langle\alpha_1,\alpha_2;\sigma,\tau|(\mu\bar{\alpha}_1-\sigma\bar{\alpha}_2)\\ \langle\alpha_1,\alpha_2;\sigma,\tau|(\mu b^\dagger+\sigma a)&=\langle\alpha_1,\alpha_2;\sigma,\tau|(\mu\bar{\alpha}_2+\sigma\bar{\alpha}_1).\end{aligned}\tag{19}$$

In spite of the fact that $\langle\alpha_1\alpha_2;\sigma,\tau|$ and $|\alpha_1\alpha_2;\mu,\nu\rangle$ are not Hermitian conjugates, they do in fact satisfy the completeness relation which may be shown by using the IWOP technique

$$\int d\bar{\alpha}_1\,d\alpha_1\int d\bar{\alpha}_2\,d\alpha_2|\alpha_1,\alpha_2;\mu,\nu\rangle\langle\alpha_1,\alpha_2;\sigma,\tau|$$
$$=\mu\tau\int\prod_{i=1}^{2}d\bar{\alpha}_i\,d\alpha_i\,:\exp\{-\bar{\alpha}_1\alpha_1-\bar{\alpha}_2\alpha_2+(a^\dagger-\sigma\tau^{-1}b)\alpha_1$$
$$+\bar{\alpha}_1(a-\nu\mu^{-1}b^\dagger)+(\sigma\tau^{-1}a+b^\dagger)\alpha_2+\bar{\alpha}_2(\nu\mu^{-1}a^\dagger+b)$$
$$+\nu\mu^{-1}(\bar{\alpha}_1\bar{\alpha}_2+a^\dagger b^\dagger)-\sigma\tau^{-1}(ab+\alpha_1\alpha_2)-a^\dagger a-b^\dagger b\}:=1$$

4. Application of the fermonic similarity transformation

As an example of the application of V we try to disentangle the operator

$$Y \equiv \exp[Aa^\dagger b^\dagger + Bab + C(a^\dagger a + b^\dagger b)]$$

where A, B and C are given constants. Performing the transformation VYV^{-1} and using (11), (12) we have

$$\begin{aligned} VYV^{-1} = \exp\{ & a^\dagger b^\dagger (A\tau^2 + B\nu^2 - 2C\tau\nu) + ab(2c\sigma\mu + B\mu^2 + A\sigma^2) \\ & + (a^\dagger a + b^\dagger b)[c(\tau\mu - \nu\sigma) + A\tau\sigma - B\mu\nu] \\ & + 2C\sigma\nu + B\mu\nu - A\sigma\tau\}. \end{aligned} \tag{21}$$

Choosing μ, ν, σ, τ to eliminate $a^\dagger b^\dagger$ and ab terms in (21), e.g. letting

$$\mu\tau + \sigma\nu = 1 \qquad 1 - 2\sigma\nu = 2C\mu\nu/A \qquad A\sigma\tau + B\mu\nu = 0 \tag{22}$$

we obtain

$$VYV^{-1} = \exp\{2D\mu\nu(a^\dagger a + b^\dagger b) + C - 2D\mu\nu\} \qquad D \equiv (C^2 - AB)/A. \tag{23}$$

As a result of (23), (14) and (22) we may express Y as

$$Y = \exp(C - 2D\mu\nu) \int \mathrm{d}\bar{\alpha}_1 \, \mathrm{d}\alpha_1 \int \mathrm{d}\bar{\alpha}_2 \, \mathrm{d}\alpha_2 \; V^{-1} \exp[2D\mu\nu(a^\dagger a + b^\dagger b)] |\alpha_1\alpha_2\rangle\langle\alpha_1\alpha_2| V \tag{24}$$

where $\langle\alpha_1\alpha_2|V$ and $V^{-1}\exp[2D\mu\nu(a^\dagger a + b^\dagger b)]|\alpha_1\alpha_2\rangle$ can be derived by using (9) and (15)

$$\langle\alpha_1\alpha_2| V = \langle 00|(-\mu) \exp\left[\frac{\nu}{\mu}\bar{\alpha}_1\bar{\alpha}_2 - \tfrac{1}{2}(\bar{\alpha}_1\alpha_1 + \bar{\alpha}_2\alpha_2) - \frac{1}{\mu}(\bar{\alpha}_1 a + \bar{\alpha}_2 b) + \frac{\sigma}{\mu} ab\right] \tag{25}$$

$$\begin{aligned} & V^{-1}\exp[2D\mu\nu(a^\dagger a + b^\dagger b)]|\alpha_1\alpha_2\rangle \\ & \quad = (-\tau)\exp\Big[-\tfrac{1}{2}(\bar{\alpha}_2\alpha_2 + \bar{\alpha}_1\alpha_1) - \frac{\sigma}{\tau}\alpha_1\alpha_2\exp(4D\mu\nu) \\ & \qquad - \frac{1}{\tau}\exp(2D\mu\nu)(a^\dagger a_1 + b^\dagger\alpha_2) - \frac{\nu}{\tau}a^\dagger b^\dagger\Big]|00\rangle. \end{aligned} \tag{26}$$

Substituting (25) and (26) into (24) and performing the integration yields Y's disentangling

$$\begin{aligned} Y = \exp(C - 2D\mu\nu) & \int \mathrm{d}\bar{\alpha}_1 \, \mathrm{d}\alpha_1 \, \mathrm{d}\bar{\alpha}_2 \, \mathrm{d}\alpha_2 \; \mu\tau \\ & \times :\exp\{-\bar{\alpha}_1\alpha_1 - \bar{\alpha}_2\alpha_2 - \tau^{-1}(a^\dagger\alpha_1 + b^\dagger\alpha_2)\,\mathrm{e}^{2D\mu\nu} \\ & - \mu^{-1}(\bar{\alpha}_1 a + \bar{\alpha}_2 b) - \sigma\tau^{-1}\alpha_1\alpha_2\,\mathrm{e}^{4D\mu\nu} + \nu\mu^{-1}\bar{\alpha}_1\bar{\alpha}_2 + \sigma\mu^{-1}ab \\ & - \nu\tau^{-1}a^\dagger b^\dagger - a^\dagger a - b^\dagger b\}: \\ = F & \exp(C - 2D\mu\nu)\exp[a^\dagger b^\dagger \mu\nu(\mathrm{e}^{4D\mu\nu} - 1)F^{-1}] \\ & \times :\exp[(a^\dagger a + b^\dagger b)(\mathrm{e}^{2D\mu\nu}F^{-1} - 1)]: \exp[ba\sigma\tau(\mathrm{e}^{4D\mu\nu} - 1)F^{-1}] \\ F & \equiv \mu\tau + \sigma\nu\exp(4D\mu\nu) \end{aligned} \tag{27}$$

The method used here can be generalized to bosonic case.

In summary, we have derived normally ordered V and V^{-1} which engender fermionic similarity transformations. We have also shown that the kets and bras produced by non-unitary operators V and V^{-1} can make up the completeness relation. Hence this work is an addenum to [1].

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